On the semitopological locally compact α-bicyclic monoid

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Definition

A semigroup S is called an *inverse semigroup* if for every a in S there exists an unique element a^{-1} in S such that

$$aa^{-1}a = a \qquad \text{and} \qquad a^{-1}aa^{-1} = a^{-1}.$$

Definition

A semigroup S is said to be:

- *bisimple* if S contains only one $\mathscr{D}$-class;
- ω^α*-semigroup* if the semigroup of idempotents of S is isomorphic to the semilattice $(\omega^\alpha, \max)$.

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A *topological* (*inverse*) *semigroup* is a topological space together with a continuous semigroup operation (and an inversion, respectively).

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The bicyclic monoid $\mathscr{B}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. It could be represented as a set $\mathbb{N}\times\mathbb{N}$ endowed with the following binary operation:

$$(a,b)\cdot(c,d)=\begin{cases}(a+c-b,d), & \text{if } b\leq c;\\ (a,d+b-c), & \text{if } b>c.\end{cases}$$

Definition

By the α-bicyclic monoid $\mathcal{B}_\alpha$ we denote the set $\omega^\alpha\times\omega^\alpha$ endowed with the following binary operation:

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Definition

Let $(S,\star)$ be an arbitrary semigroup. The Bruck extension $B(S)$ of a semigroup S is a set $\mathbb{N}\times S\times\mathbb{N}$ endowed with the following semigroup operation:

$$(a,s,b)\cdot(c,t,d)=\begin{cases}(a+c-b,t,d), & \text{if } \ b<c;\\ (a,s,d+b-c), & \text{if } \ b>c;\\ (a,s\star t,d), & \text{if } \ b=c.\end{cases}$$

Definition

Let $(S,\star)$ be a monoid and θ be a homomorphism from S to the group of units $G(S)$ of S. The Bruck-Reilly extension $BR(S)$ of a semigroup S is a set $\mathbb{N}\times S\times\mathbb{N}$ endowed with the following semigroup operation:

$$(a,s,b)\cdot(c,t,d)=\begin{cases}(a+c-b,\theta^{c-b}(s)\star t,d), & \text{if } \ b<c;\\ (a,s\star\theta^{b-c}(t),d+b-c), & \text{if } \ b>c;\\ (a,s\star t,d), & \text{if } \ b=c.\end{cases}$$

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Proposition (Reilly, 1966).

Each bisimple regular ω-semigroup is isomorphic to the Bruck–Reilly extension of some group.

Proposition (Hogan, 1973).

Each bisimple ω^α-semigroup is isomorphic to the product $G \times \mathcal{B}_\alpha$ of the α-bicyclic semigroup and some group G, endowed with a following semigroup operation:

$$(s,(a,b))\cdot(t,(c,d)) = \begin{cases} (st,(a,b)(c,d)) & \text{if } b = c; \\ (s|(b-c)^c|^{-1}(t\theta'_{b-c})|(b-c)^d|,(a,b)(cd)), & \text{if } b > c; \\ (|(c-b)^a|^{-1}(s\theta'_{c-b})|(c-b)^b|t,(a,b)(cd)), & \text{if } b < c. \end{cases}$$

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Proposition (Eberhart, Selden, 1969).

The only Hausdorff topology on $\mathscr{C}(p,q)$, which makes $\mathscr{C}(p,q)$ into a topological semigroup is the discrete topology.

Proposition (Bertman, West, 1976).

The only Hausdorff topology on $\mathscr{C}(p,q)$, which makes $\mathscr{C}(p,q)$ into a semitopological semigroup is the discrete topology.

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There exists a non-discrete Hausdorff topology on $\mathcal{B}_2$, which makes $\mathcal{B}_2$ into a topological inverse semigroup.

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Proposition 1

For each ordinal α the α-bicyclic monoid $\mathcal{B}_\alpha$ is isomorphic to a semigroup of all order isomorphisms between the principal upper sets of the ordinal ω^α.

Problem (Nowak, 7 hours ago)

Are you a self-similar?

Proposition 2

For each ordinal α the $\alpha+1$-bicyclic semigroup $\mathcal{B}_{\alpha+1}$ is isomorphic to the Bruck extension of the α-bicyclic semigroup $\mathcal{B}_\alpha$.

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Let $\mathcal{B}_\alpha$ be a Hausdorff semitopological semigroup and $(a, b) \in \mathcal{B}_\alpha$ be an arbitrary element. If either a or b is an non-limit ordinal then (a, b) is an isolated point in $\mathcal{B}_\alpha$.

Proposition 2

For each ordinal $\alpha < \omega + 1$ the only Hausdorff locally compact topology which makes $\mathcal{B}_\alpha$ into a topological semigroup is the discrete topology.

Proposition 3

There exists an example of a non-discrete Hausdorff locally compact topological inverse $\omega + 1$-bicyclic monoid $\mathcal{B}_{\omega+1}$.

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Proposition 4

For each positive integer n there exist exactly n distinct Hausdorff locally compact topologies which make $\mathcal{B}_n$ into a semitopological semigroup.

Proposition 5

The set of all Hausdorff locally compact topologies which make $\mathcal{B}_\omega$ into a semitopological semigroup is countable.

Proposition 6

For each ordinal $\alpha < \omega + 1$ the set of all Hausdorff locally compact topologies which make $\mathcal{B}_\alpha$ into a semitopological semigroup is a linearly ordered lattice.

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Thank You for attention!